

*P-Finsler spaces with vanishing Douglas tensor

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Abstract. The purpose of the present paper is to prove that a *P-Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three.

1. Introduction

Let $F^n(M^n, L)$ be an n -dimensional Finsler space, where M^n is a connected differentiable manifold of dimension n and $L(x, y)$ is the fundamental function defined on the manifold $T(M) \setminus 0$ of nonzero tangent vectors. Let us consider a geodesic curve $x^i = x^i(t)$,¹ ($t_0 \leq t \leq t_1$). The system of differential equations for geodesic curves of F^n with respect to canonical parameter t is given by

$$\frac{d^2 x^i}{dt^2} = -2G^i(x, y), \quad y^i = \frac{dx^i}{dt},$$

where

$$G^i = \frac{1}{4} g^{ir} \left(y^s \left(\frac{\partial L^2_{(r)}}{\partial x^s} \right) - \frac{\partial L^2}{\partial x^r} \right),$$
$$g_{ij} = \frac{1}{2} L^2_{(i)(j)}, \quad (i) = \frac{\partial}{\partial y^i}, \quad \text{and} \quad (g^{ij}) = (g_{ij})^{-1}.$$

The Berwald connection coefficients $G^i_j(x, y)$, $G^i_{jk}(x, y)$ can be derived from the function G^i , namely $G^i_j = G^i_{(j)}$ and $G^i_{jk} = G^i_{j(k)}$. The Berwald covariant derivative with respect to the Berwald connection can be written as

$$(1) \quad T^i_{j;k} = \partial T^i_j / \partial x^k - T^i_{j(r)} G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

(Throughout the present paper we shall use the terminology and definitions described in Matsumoto's monograph [6].)

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¹ The Roman indices run over the range $1, \dots, n$.

2. Douglas tensor, Randers metric, *P-space

Let us consider two Finsler space $F^n(M^n, L)$ and $\bar{F}^n(M^n, \bar{L})$ on a common underlying manifold M^n . A diffeomorphism $F^n \rightarrow \bar{F}^n$ is called geodesic if it maps an arbitrary geodesic of F^n to a geodesic of \bar{F}^n . In this case the change $L \rightarrow \bar{L}$ of the metric is called projective. It is well-known that the mapping $F^n \rightarrow \bar{F}^n$ is geodesic iff there exist a scalar field $p(x, y)$ satisfying the following equation

$$(2) \quad \bar{G}^i = G^i + p(x, y)y^i, \quad p \neq 0.$$

The projective factor $p(x, y)$ is a positive homogeneous function of degree one in y . From (2) we obtain the following equations

$$(3) \quad \bar{G}_j^i = G_j^i + p\delta_j^i + p_j y^i, \quad p_j = p_{(j)},$$

$$(4) \quad \bar{G}_{jk}^i = G_{jk}^i + p_j \delta_k^i + p_k \delta_j^i + p_{jk} y^i, \quad p_{jk} = p_{j(k)},$$

$$(5) \quad \bar{G}_{jkl}^i = G_{jkl}^i + p_{jk} \delta_l^i + p_{jl} \delta_k^i + p_{kl} \delta_j^i + p_{jkl} y^i, \quad p_{jkl} = p_{j(kl)}.$$

Substituting $p_{ij} = (\bar{G}_{ij} - G_{ij}) / (n+1)$ and $p_{ijk} = (\bar{G}_{ij(k)} - G_{ij(k)}) / (n+1)$ into (5) we obtain the so called Douglas tensor which is invariant under geodesic mappings, that is

$$(6) \quad D_{jkl}^i = G_{jkl}^i - (y^i G_{jk(l)} + \delta_j^i G_{kl} + \delta_k^i G_{jl} + \delta_l^i G_{jk}) / (n+1),$$

which is invariant under geodesic mappings, that is

$$(7) \quad D_{jkl}^i = \bar{D}_{jkl}^i.$$

We now consider some notions and theorems for special Finsler spaces.

Definition 1. ([1]) In an n -dimensional differentiable manifold M^n a Finsler metric $L(x, y) = \alpha(x, y) + \beta(x, y)$ is called Randers metric, where $\alpha(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric in M^n and $\beta(x, y) = b_i(x)y^i$ is a differential 1-form in M^n . The Finsler space $F^n = (M^n, L) = \alpha + \beta$ with Randers metric is called Randers space.

Definition 2. ([1]) The Finsler metric $L = \alpha^2 / \beta$ is called Kropina metric. The Finsler space $F^n = (M^n, L) = \alpha^2 / \beta$ with Kropina metric is called Kropina space.

Definition 3. ([1], [6]) A Finsler space of dimension $n > 2$ is called *C-reducible*, if the tensor $C_{ijk} = \frac{1}{2}g_{ij(k)}$ can be written in the form

$$(8) \quad C_{ijk} = \frac{1}{n+1} (h_{ij}C_k + h_{ik}C_j + h_{jk}C_i),$$

where $h_{ij} = g_{ij} - l_i l_j$ is the angular metric tensor and $l_i = L_{(i)}$.

Theorem 1. ([7]) A Finsler space F^n , $n \geq 3$, is *C-reducible* iff the metric is a Randers metric or a Kropina metric.

Definition 4. ([4], [5]) A Finsler space F^n is called **P-Finsler space*, if the tensor $P_{ijk} = \frac{1}{2}g_{ij;k}$ can be written in the form

$$(9) \quad P_{ijk} = \lambda(x, y)C_{ijk}.$$

Theorem 2. ([4]) For $n > 3$ in a *C-reducible *P-Finsler space* $\lambda(x, y) = k(x)L(x, y)$ holds and $k(x)$ is only the function of position.

3. *P-Randers space with vanishing Douglas tensor

Definition 5. ([3]) A Finsler space is said to be of Douglas type or Douglas space, iff the functions $G^i y^j - G^j y^i$ are homogeneous polynomials in (y^i) of degree three.

Theorem 3. ([3]) A Finsler space is of Douglas type iff the Douglas tensor vanishes identically.

Theorem 4. ([5]) For $n > 3$, in a *C-reducible *P-Finsler space* $D_{jkl}^i = 0$ holds.

If we consider a Randers change

$$\bar{L}(x, y) \rightarrow L(x, y) + \beta(x, y),$$

where $\beta(x, y)$ is a closed one-form, then this change $\bar{L} \rightarrow L$ is projective.

Definition 6. ([1]) A Finsler space is called Landsberg space if the condition $P_{ijk} = 0$ holds.

Theorem 5. ([2]) If there exist a Randers change with respect to a projective scalar $p(x, y)$ between a Landsberg and a *P-Finsler space (fulfilling the condition $\bar{P}_{ijk} = p(x, y)\bar{C}_{ijk}$), then $p(x, y)$ can be given by the equation

$$(10) \quad p(x, y) = e^{\varphi(x)}\bar{L}(x, y).$$

It is well-known that the Riemannian space is a special case of the Landsberg space. In a Riemannian space we have $D_{jkl}^i = 0$, and a *P -Randers space with a closed one-form $\beta(x, y)$ is a Finsler space with vanishing Douglas tensor

Theorem 6. ([3]) *A Randers space is a Douglas space iff $\beta(x, y)$ is a closed form. Then*

$$(11) \quad 2G^i = \gamma_{jk}^i y^j y^k + \frac{r_{lm} y^l y^m}{\alpha + \beta} y^i,$$

where $\gamma_{jk}^i(x)$ is the Levi-Civita connection of a Riemannian space, r_{lm} is equal to $b_{i;j}$ hence r_{lm} depends only on position.

From the Theorem 6. and (10) follows that

$$\frac{r_{lm} y^l y^m}{\alpha + \beta} = e^{\varphi(x)} (\alpha + \beta)$$

that is

$$\frac{r_{lm} y^l y^m}{\overline{L}} = e^{\varphi(x)} \overline{L}.$$

From the last equation we obtain

$$r_{lm} y^l y^m = e^{\varphi(x)} \overline{L}^2.$$

Differentiating twice this equation by y^l and y^m we get

$$b_{i;j} = e^{\varphi(x)} \overline{g}_{ij}.$$

This means that the metrical tensor \overline{g}_{ij} depends only on x , so we get the following

Theorem. *A *P -Randers space with vanishing Douglas tensor is a Riemannian space if the dimension is greater than three.*

4. Further possibilities

From Theorem 1, Theorem 4 and our Theorem follows that only the *P -Kropina spaces can be *P - C reducible spaces with vanishing Douglas tensor which are different from Riemannian spaces. We would like to investigate this latter case in a forthcoming paper.

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